MAT Syllabus Practice Solutions

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Polynomials

- Solve $x^2 - x - 1 = 0$

  The quadratic formula gives $x = \frac{1\pm\sqrt{5}}{2}$.

- Solve $x^4 - x^2 - 1 = 0$

  Write $y = x^2$ to get a quadratic for y. This is the quadratic above for y, so $x^2 = \frac{1\pm\sqrt{5}}{2}$. But $x^2 \geq 0$ so $x = \pm\sqrt{\frac{1+\sqrt{5}}{2}}$.

- Write $x^2 + 4x + 3$ in the form $(x + a)^2 + b$

  $(x + 2)^2 - 1$

- How many real solutions does $x^2 + bx + 1 = 0$ have? Find the different cases in terms of b.

  The discriminant, $b^2 - 4$, is positive if $b > 2$ or $b < -2$, negative if $-2 < b < 2$ and zero if $b = \pm 2$. So there are two real solutions if $b > 2$ or if $b < -2$, one real solution if $b = \pm 2$ and no real solutions otherwise.

- Factorise $x^2 + 4x + 3$

  $(x + 1)(x + 3)$

- Let $p(x) = x^3 - 13x^2 - 65x - 51$. Check that $p(17) = 0$. Factorise $p(x)$.

  $p(17) = 17^3 - 13 \times 17^2 - 65 \times 17 - 51 = 17\left(17^2 - 13 \times 17 - 65 - 3\right) = 17^2(17 - 13 - 4) = 0$. So $(x - 17)$ is a factor. Polynomial division gives $p(x) = (x - 17)(x^2 + 4x + 3)$, so $p(x) = (x - 17)(x + 1)(x + 3)$.

Algebra

- Solve the simultaneous equations $x + y = 1$ and $x - y = 3$.

  $x = 2$ and $y = -1$.

- For which values of x is it true that $x^2 + 4x + 3 > 0$?

  $x > -1$ or $x < -3$.

- Expand $(2x + 3)^3$

  $8x^3 + 36x^2 + 54x + 27$

- I've got four playing cards; the ace and king of clubs, and the ace and king of hearts. I shuffle the cards together and deal them out left to right. What's the probability that the kings and aces alternate? (they alternate if they are either arranged as $AKAK$ or $KAKA$)

  There are 24 possible orders for the cards. Eight of these have alternating kings and aces, so the probability is 1/3.

Differentiation

- Differentiate x^{17} with respect to x.

  $17x^{16}$

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- Differentiate $\sqrt{x}$ with respect to x.

  $\frac{1}{2\sqrt{x}}$

- Differentiate e^{3x} with respect to x.

  $3e^{3x}$

- Differentiate $2e^{-x} - x^2$ with respect to x.

  $-2e^{-x} - 2x$

- Find the tangent to the curve $y = e^x + 1$ at $x = 2$.

  $y = e^2(x-2) + e^2 + 1$

- Find the normal to the parabola $y = x^2$ at $x = 3$.

  $y = -\frac{1}{6}(x-3) + 9$

- Find the turning points of the curve $y = x^4 - 2x^3 + x^2$. Identify whether the turning points are maxima or minima.

  Turning points at $x = 0$ (minimum), $x = \frac{1}{2}$ (maximum), $x = 1$ (minimum).

- For which values of x is $y = x^4 - 2x^3 + x^2$ increasing? For which values of x is it decreasing?

  Increasing for $0 < x < \frac{1}{2}$ and for $1 < x$. Decreasing for $x < 0$ and for $\frac{1}{2} < x < 1$.

- Two points A and B are on the curve $y = x^3 + x^2 + x + 1$. A is held fixed at $(1, 4)$. The point B is moved along the curve towards A. What happens to the line through A and B?

  The tangent at A is $y = 6x - 2$. If the line AB has equation $y = mx + c$ say, then m gets closer and closer to 6 and c gets closer and closer to -2.

Integration

- Suppose that the derivative of a polynomial $p(x)$ with respect to x is $q(x)$. Find $\displaystyle\int q(x)\,\mathrm{d}x$.

  $p(x) + c$ where c is a constant

- Find the area enclosed by the polynomial $x^2 + 4x + 3 = 0$ and the x-axis.

  $\frac{4}{3}$

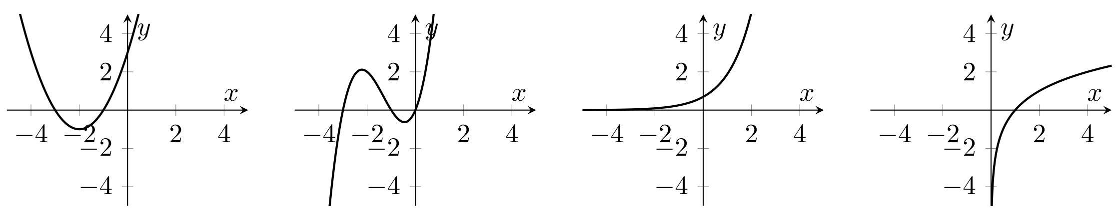
- Find $\displaystyle\int_{-1}^{1} 1 + x + x^2 + x^3 + x^4 + x^5 + x^6\,\mathrm{d}x$

  Note that $\int_{-1}^{1} x^a\,\mathrm{d}x = 0$ for a odd. The integral is $2\big(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}\big) = \frac{352}{105}$.

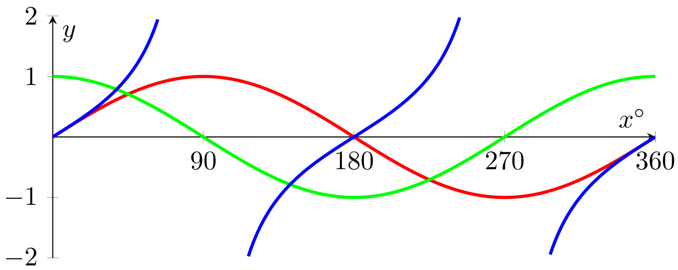
Graphs

- Sketch graphs of

  $$y = x^2 + 4x + 3, \quad y = x^3 + 4x^2 + 3x, \quad y = 2^x, \quad y = \log_2 x \quad \text{on separate axes.}$$



- Sketch graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ on the same axes.



Logarithms and powers

- Simplify $\log 3 + \log 4$ into a single term.

  $\log 12$

- Expand $(e^x + e^{-x})(e^x + e^{-x})$

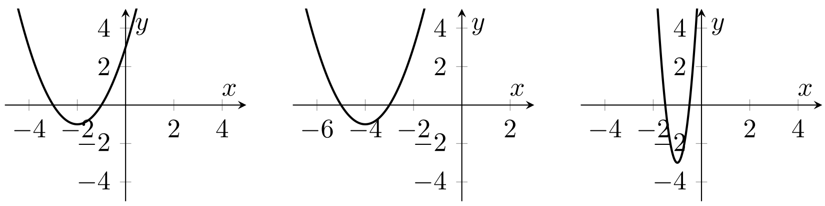
  $e^{2x} + 2 + e^{-2x}$.

- Solve $2^x = 3$.

  $x = \log_2 3$

Transformations

- Let $f(x) = x^2 + 4x + 3$. If you didn't sketch a graph of this before, sketch one now.
- Sketch a graph of $y = f(x + 2)$.
- Sketch a graph of $y = 3f(2x)$.



Geometry

- Add the vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

  $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$

- Find the equation of the line through $(1, 0)$ and $(0, -1)$.

  $y = x - 1$

- Find the equation of the line through $(1, 2)$ with gradient 3.

  $y = 3(x - 1) + 2 = 3x - 1$

- A circle has centre $(-1, 4)$ and radius 3. Write down an equation for the circle.

  $(x + 1)^2 + (y - 4)^2 = 9$

- What's the area of this circle?

  9π

- Points A and B lie on a circle with centre O and radius 1. The angle $\angle AOB$ is $120°$. Find the length of the arc between A and B. Find the area enclosed by that arc and the radii OA and OB.

  It's a third of a circle, so the arc length is $2\pi/3$ and the area is $\pi/3$.

Trigonometry

- Solve $\sin x = \frac{1}{2}$.

  $x = 30° + n \times 360°$, or $x = 150° + n \times 360°$, for any whole number n.

- Solve $\tan x = 1$.

  $x = 45° + n \times 180°$ for any whole number n

- Write $\cos^4 x + \cos^2 x$ in terms of $\sin x$.

  $(1 - \sin^2 x)^2 + (1 - \sin^2 x) = 2 - 3\sin^2 x + \sin^4 x$.

- Simplify $\cos(450° - x)$

  $\sin x$

- A triangle ABC has side lengths $AB = 3$ and $BC = 2$, and the angle $\angle ABC = 120°$. Find the remaining side length AC, the area of the triangle, and an expression for $\sin \angle BCA$.

  Cosine rule; $AC = \sqrt{19}$. The area of the triangle is $3\sqrt{3}/2$. Sine rule; $\sin \angle BCA = (3\sqrt{3}/2\sqrt{19})$

Sequences and series

- A sequence is defined by $a_0 = 1$, $a_1 = 1$, $a_2 = 1$, and

  $$a_n = a_{n-1} + a_{n-2} + a_{n-3} \quad \text{for } n \geq 3.$$

  Find a_{10}.

  $a_3 = 3$, $a_4 = 5$, $a_5 = 9$, $a_6 = 17$, $a_7 = 31$, $a_8 = 57$, $a_9 = 105$, $a_{10} = 193$.

- A sequence has first term 3 and each subsequent term is 5 more than the previous term. Find the sum of the first four terms.

  $4 \times 3 + \frac{4 \times 3}{2} \times 5 = 42$

- A sequence has first term 4 and each subsequent term is 6 times more than the previous term. Find the sum of the first four terms.

  $4\left(1 + 6 + 6^2 + 6^3\right) = 4\frac{6^4 - 1}{6 - 1} = 4\frac{1295}{5} = 4 \times 259 = 1036$.

- When does the sum $1 + x^3 + x^6 + x^9 + x^{12} + \ldots$ converge? Simplify it in the case that it converges.

  Converges when $-1 < x < 1$. In that case, it converges to $1/(1 - x^3)$.